

Tomographic Representation of Quantum Mechanics and Statistical Physics

Olga V. Man'ko*

Аннотация

A review of the photon-number tomography and symplectic tomography as examples of star-product quantization is presented. The classical statistical mechanics is considered within the framework of the tomographic representation.

1 INTRODUCTION

In quantum mechanics, the state is described by the wave function or density matrix (density operator). In classical mechanics, the state of system with fluctuations is described by the probability-distribution function. The quantum and classical natures are described using different objects – operators and functions. But for more easy understanding the nature of systems which has classical and quantum parts, it is necessary to have the same language for both fields. So, we need the representation in which classical and quantum states are described by the same object – the probability distribution function. For this purpose, the tomographic probability representation of quantum mechanics for continuous variables called symplectic tomography was introduced in [1–3]. The probability representation of quantum mechanics was extended to the case of discrete variables in infinite domain (photon-number tomography) in [3–5]. The name photon-number tomography was given in [3] for elucidating the physical meaning of measuring the quantum state by means of measuring the number of photons (i.e., photon statistics). Photon-number tomography is the method to reconstruct the density operator of quantum state using measurable probability-distribution function (photon statistics) called tomogram. The tomographic-probability representation of classical states was introduced in [6–9]. In the probability representation, the quantum states and the classical states are described by the same objects – tomograms. Tomograms are positive measurable probability-distribution functions of random variables. In order to describe observables by functions instead of operators in quantum mechanics, the quantization based on star-product of functions is used. In [10, 11] it was shown that the symplectic-tomography scheme is a new example of quantization based on star-product of functions – symbols of operators. In [12, 13] it was shown that photon-number tomography is an example of the star-product quantization.

The aim of this paper is to present a review of the tomographic approaches and their connection with star-product quantization scheme.

*P. N. Lebedev Physical Institute, Leninskii Prospect 53, Moscow 119991, Russia, e-mail: omanko@sci.lebedev.ru

2 GENERAL STAR-PRODUCT SCHEME

Following [10,11], we consider a density operator $\hat{\rho}$ acting in a given Hilbert space. Let us suppose that we have a set of operators $\hat{\mathcal{U}}(\mathbf{x})$ acting in the Hilbert space H , where the n -dimensional vector $\mathbf{x} = (x_1, x_2, \dots, x_n)$ labels the particular operator in the set. We construct the c -number function $w_{\hat{\rho}}(\mathbf{x})$ using the definition

$$w_{\hat{\rho}}(\mathbf{x}) = \text{Tr}(\hat{\rho}\hat{\mathcal{U}}(\mathbf{x})). \quad (1)$$

The function $w_{\hat{\rho}}(\mathbf{x})$ is called tomogram, it is the symbol of density operator $\hat{\rho}$, and the operators $\hat{\mathcal{U}}(\mathbf{x})$ are dequantizers [14]. We suppose that this relation has the inverse, i.e., there exists the set of operators $\hat{\mathcal{D}}(\mathbf{x})$ acting in the Hilbert space such that

$$\hat{\rho} = \int w_{\hat{\rho}}(\mathbf{x})\hat{\mathcal{D}}(\mathbf{x}) d\mathbf{x}. \quad (2)$$

The operators $\hat{\mathcal{D}}(\mathbf{x})$ are quantizers [14]. Formulas (1) and (2) are selfconsistent, if one has the following property of the quantizers and dequantizers:

$$\text{Tr}[\hat{\mathcal{U}}(\mathbf{x})\hat{\mathcal{D}}(\mathbf{x}')] = \delta(\mathbf{x} - \mathbf{x}'). \quad (3)$$

Relations (1) and (2) determine the invertible map of the density operator $\hat{\rho}$ onto function – tomogram $w_{\hat{\rho}}(\mathbf{x})$. We introduce the associative product (star-product) of two tomograms $w_{\hat{\rho}_1}(\mathbf{x})$ and $w_{\hat{\rho}_2}(\mathbf{x})$ corresponding to two density operators $\hat{\rho}_1$ and $\hat{\rho}_2$, respectively, by the relations

$$w_{\hat{\rho}_1\hat{\rho}_2}(\mathbf{x}) = w_{\hat{\rho}_1}(\mathbf{x}) * w_{\hat{\rho}_2}(\mathbf{x}) = \text{Tr}(\hat{\rho}_1\hat{\rho}_2\hat{\mathcal{U}}(\mathbf{x})). \quad (4)$$

The map provides the nonlocal product of two tomograms (star-product)

$$w_{\hat{\rho}_1}(\mathbf{x}) * w_{\hat{\rho}_2}(\mathbf{x}) = \int w_{\hat{\rho}_1}(\mathbf{x}'')w_{\hat{\rho}_2}(\mathbf{x}')K(\mathbf{x}'', \mathbf{x}', \mathbf{x}) d\mathbf{x}' d\mathbf{x}''.$$

The kernel of the star-product is linear with respect to the dequantizer and nonlinear in the quantizer operator

$$K(\mathbf{x}'', \mathbf{x}', \mathbf{x}) = \text{Tr}[\hat{\mathcal{D}}(\mathbf{x}'')\hat{\mathcal{D}}(\mathbf{x}')\hat{\mathcal{U}}(\mathbf{x})].$$

The associativity condition for tomograms means that the kernel of star-product of tomograms $K(\mathbf{x}'', \mathbf{x}', \mathbf{x})$ satisfies the nonlinear equation [14]

$$\int K(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y})K(\mathbf{y}, \mathbf{x}_3, \mathbf{x}_4)d\mathbf{y} = \int K(\mathbf{x}_1, \mathbf{y}, \mathbf{x}_4)K(\mathbf{x}_2, \mathbf{x}_3, \mathbf{y})d\mathbf{y}. \quad (5)$$

3 SYMPLECTIC TOMOGRAPHY

Now we consider the symplectic-tomography scheme [1] as an example of the star-product quantization, following [10,11]. In this scheme, the tomographic symbol of the density operator $\hat{\rho}$ called symplectic tomogram $w_{\hat{\rho}}(\mathbf{x})$ is obtained due to formula (1) by means of the dequantizer

$$\hat{\mathcal{U}}(X, \mu, \nu) = \delta(X\hat{1} - \mu\hat{q} - \nu\hat{p}),$$

where vector $\mathbf{x} = (X, \mu, \nu)$ has the coordinates which are real numbers and $\hat{1}$ is identity operator. The operators \hat{q} and \hat{p} are the position and momentum operators, respectively. The symplectic tomogram $w(X, \mu, \nu)$ depends on the two extra real parameters μ and ν . The physical meaning of the parameters μ and ν is that they describe an ensemble of rotated and scaled reference frames in which the position X is measured. For $\mu = \cos \varphi$ and $\nu = \sin \varphi$, the symplectic tomogram coincides with the marginal distribution for the homodyne-output variable used in optical tomography [15, 16].

Symplectic tomogram can be determined as the expectation value of the delta-function calculated with the help of the density operator

$$w(X, \mu, \nu) = \langle \delta(\mu\hat{q} + \nu\hat{p} - X) \rangle. \quad (6)$$

The symplectic tomogram is nonnegative function

$$w(X, \mu, \nu) \geq 0,$$

and it is normalized with respect to the variable X

$$\int w(X, \mu, \nu) dX = 1.$$

The tomogram is a homogenous function

$$w(\lambda X, \lambda\mu, \lambda\nu) = |\lambda|^{-1} w(X, \mu, \nu).$$

The density operator of the state can be reconstructed in the symplectic tomography scheme, in view of (2), with the help of quantizer

$$\hat{\mathcal{D}}(X, \mu, \nu) = \frac{1}{2\pi} \exp(iX\hat{1} - i\nu\hat{p} - i\mu\hat{q}).$$

Since the density operator determines completely the quantum state of a system and, on the other hand, the density operator itself is completely determined by symplectic tomogram, one can use for describing quantum states symplectic tomograms (positive and normalized) which are probability-distribution functions analogous to the classical ones. The quantum state is given if the position probability distribution $w(X, \mu, \nu)$ in an ensemble of rotated and squeezed reference frames in phase space is given. The information contained in symplectic tomogram $w(X, \mu, \nu)$ is overcomplete. To determine the quantum state, it is enough to know the values of symplectic tomogram of the state for arguments satisfying the following condition: $(\mu^2 + \nu^2 = 1)$, where $\mu = \cos \varphi$.

The kernel of star-product of two symplectic tomograms of density operators $\hat{\rho}_1$ and $\hat{\rho}_2$ has the following form [10]:

$$K(X_1, \mu_1, \nu_1, X_2, \mu_2, \nu_2, X, \mu, \nu) = \frac{\delta(\mu(\nu_1 + \nu_2) - \nu(\mu_1 + \mu_2))}{4\pi^2} \times \exp\left(\frac{i}{2}\left\{(\nu_1\mu_2 - \nu_2\mu_1) + 2X_1 + 2X_2 - \frac{2(\nu_1 + \nu_2)X}{\nu}\right\}\right).$$

4 PHOTON-NUMBER TOMOGRAPHY

The photon-number tomogram defined by the relation

$$\omega(n, \alpha) = \langle n | \hat{D}(\alpha) \hat{\rho} \hat{D}^{-1}(\alpha) | n \rangle \quad (7)$$

is the function of integer photon number n and complex number $\alpha = \text{Re } \alpha + i \text{Im } \alpha$, where $\hat{\rho}$ is the state density operator and $\hat{D}(\alpha)$ is the Weyl displacement operator

$$\hat{D}(\alpha) = \exp(\alpha \hat{a}^\dagger - \alpha^* \hat{a}).$$

The photon-number tomogram is a symbol of the density operator

$$\omega(n, \alpha) = \text{Tr} [\hat{\rho} \hat{\mathcal{U}}(\mathbf{x})], \quad (8)$$

where dequantizer $\hat{\mathcal{U}}(\mathbf{x})$ reads

$$\hat{\mathcal{U}}(\mathbf{x}) = \hat{D}(\alpha) |n\rangle \langle n| \hat{D}^{-1}(\alpha), \quad \mathbf{x} = (n, \alpha).$$

The density operator can be reconstructed from the photon-number tomogram with the help of the inverse formula [3–5]

$$\hat{\rho} = \sum_{n=0}^{\infty} \int \frac{4 d^2 \alpha}{\pi(1-s^2)} \left(\frac{s-1}{s+1} \right)^{(\hat{a}^\dagger + \alpha^*)(\hat{a} + \alpha) - n} \omega(n, \alpha), \quad (9)$$

where s is an arbitrary ordering parameter [17]. The quantizer in the photon-number-tomography scheme is

$$\hat{\mathcal{D}}(\mathbf{x}) = \frac{4}{\pi(1-s^2)} \left(\frac{s-1}{s+1} \right)^{(\hat{a}^\dagger + \alpha^*)(\hat{a} + \alpha) - n}. \quad (10)$$

It is known [17] that the Wigner function [18], which corresponds to the density operator $\hat{\rho}$, is given by the expression

$$W_{\hat{\rho}}(q, p) = 2 \text{Tr} [\hat{\rho} \hat{D}(\beta) (-1)^{\hat{a}^\dagger \hat{a}} \hat{D}(-\beta)], \quad (11)$$

where $\hat{D}(\beta)$ is the Weyl displacement operator with complex argument

$$\beta = \frac{1}{\sqrt{2}}(q + ip),$$

with \hat{a} and \hat{a}^\dagger being the photon annihilation and creation operators.

Let us introduce the displaced density operator

$$\hat{\rho}_\alpha = \hat{D}^{-1}(\alpha) \hat{\rho} \hat{D}(\alpha). \quad (12)$$

The Wigner function, which corresponds to the displaced density operator, is of the form

$$W_{\hat{\rho}_\alpha}(q, p) = 2 \text{Tr} [\hat{\rho}_\alpha \hat{D}(\beta) (-1)^{\hat{a}^\dagger \hat{a}} \hat{D}(-\beta)]. \quad (13)$$

By inserting the expression for the displaced density operator into (13), one arrives at

$$W_{\hat{\rho}_\alpha}(q, p) = 2 \operatorname{Tr} \left[\hat{D}^{-1}(\alpha) \hat{\rho} \hat{D}(\alpha) \hat{D}(\beta) (-1)^{\hat{a}^\dagger \hat{a}} \hat{D}(-\beta) \right]. \quad (14)$$

In view of the properties of the Weyl displacement operator

$$\begin{aligned} \hat{D}(\beta) \hat{D}(\alpha) &= \hat{D}(\beta + \alpha) \exp \left[i \operatorname{Im}(\beta \alpha^*) \right], \quad \hat{D}^{-1}(\alpha) = \hat{D}(-\alpha), \\ \hat{D}^{-1}(\alpha) \hat{D}^{-1}(\beta) &= \left(\hat{D}(\beta) \hat{D}(\alpha) \right)^{-1}, \end{aligned}$$

formula (14) can be simplified as follows:

$$W_{\hat{\rho}_\alpha}(q, p) = W_{\hat{\rho}}(q + \sqrt{2} \operatorname{Re} \alpha, p + \sqrt{2} \operatorname{Im} \alpha). \quad (15)$$

One can see that the Wigner function (13) corresponding to the displaced density operator is equal to the Wigner function (21) corresponding to the initial density operator but with displaced arguments. The photon-number tomogram is the photon distribution function (the probability to have n photons) in the state described by the displaced density operator $\hat{\rho}_\alpha$ (12), i.e.,

$$\omega(n, \alpha) = P_n(\alpha) = \langle n | \hat{\rho}_\alpha | n \rangle, \quad n = 0, 1, 2, \dots \quad (16)$$

As an example, we consider, following [19, 20], the photon-number tomogram of the Gaussian state of one-mode light described by the Wigner function of generic Gaussian form

$$W(q, p) = \frac{1}{\sqrt{\det \sigma(t)}} \exp \left(-\frac{1}{2} \mathbf{Q} \sigma^{-1}(t) \mathbf{Q}^T \right), \quad (17)$$

where $\mathbf{Q} = (p - \langle p \rangle, q - \langle q \rangle)$ and the matrix $\sigma(t)$ is a real symmetric quadrature variance matrix. The photon distribution function for one-mode mixed light was obtained explicitly in terms of the Hermite polynomials of two variables in [21]. The Hermite polynomials of two variables $H_{n_1 n_2}^{\{\mathbf{R}\}}(y_1, y_2)$, where n_1, n_2 are nonnegative integers and \mathbf{R} is a symmetric 2×2 matrix, are determined by the generating function

$$\begin{aligned} &\exp \left[-\frac{1}{2} (x_1 \ x_2) \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (y_1 \ y_2) \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right] = \\ &\sum_{n_1, n_2=0}^{\infty} \frac{x_1^{n_1} x_2^{n_2}}{n_1! n_2!} H_{n_1 n_2}^{\{\mathbf{R}\}}(y_1, y_2). \end{aligned}$$

Applying the scheme of calculations similar to the one used in [21] to our photon-number tomogram (16), we arrive at the photon-number tomogram as a function of the Hermite polynomial of two variables

$$\omega(n, \alpha) = \frac{P_0(\alpha) H_{nn}^{\{\mathbf{R}\}}(y_1(\alpha), y_2(\alpha))}{n!}, \quad (18)$$

where the matrix \mathbf{R} , which determines the Hermite polynomial, reads

$$\mathbf{R} = \frac{1}{1 + 2T + 4d} \begin{pmatrix} 2(\sigma_{pp} - \sigma_{qq} - 2i\sigma_{pq}) & 1 - 4d \\ 1 - 4d & 2(\sigma_{pp} - \sigma_{qq} + 2i\sigma_{pq}) \end{pmatrix}.$$

Here d is the determinant of real symmetric quadrature variance matrix $\sigma(t)$, i.e., $d = \sigma_{pp}\sigma_{qq} - \sigma_{pq}^2$ and T is its trace $T = \sigma_{pp} + \sigma_{qq}$. The arguments of the Hermite polynomial are

$$y_1(\alpha) = y_2^*(\alpha) = \frac{\sqrt{2}}{2T - 4d - 1} \left[(\langle q \rangle - i\langle p \rangle + \sqrt{2}\alpha^*) (T - 1) + (\sigma_{pp} - \sigma_{qq} + 2i\sigma_{pq}) (\langle q \rangle + i\langle p \rangle + \sqrt{2}\alpha) \right]. \quad (19)$$

For the state with displaced Wigner function (13), the probability to have no photons $P_0(\alpha)$ reads

$$P_0(\alpha) = \frac{2}{\sqrt{L}} \exp \left\{ -\frac{1}{L} \left[(2\sigma_{qq} + 1) (\langle p \rangle + \sqrt{2}\text{Im } \alpha)^2 + (2\sigma_{pp} + 1) (\langle q \rangle + \sqrt{2}\text{Re } \alpha)^2 \right] \right\} \\ \times \exp \left[\frac{4\sigma_{pq}}{L} (\langle p \rangle + \sqrt{2}\text{Im } \alpha) (\langle q \rangle + \sqrt{2}\text{Re } \alpha) \right], \quad (20)$$

where $L = 1 + 2T + 4d$.

5 CLASSICAL STATES

Let us consider the classical state, for example, the state of a classical particle with one degree of freedom with unit mass. We suppose that the particle's position q and momentum p fluctuate due to the interaction between the particle and some environment. The particle's state is described by a probability distribution function $f_{\text{cl}}(q, p)$, which is nonnegative $f_{\text{cl}}(q, p) \geq 0$ and normalized. Here we assumed the normalization condition for $f_{\text{cl}}(q, p)$

$$\frac{1}{2\pi} \int f_{\text{cl}}(q, p) dq dp = 1, \quad (21)$$

analogously to the normalization condition of the Wigner function $W(q, p)$.

The tomogram of this state is determined by the Radon transform of the probability distribution function $f_{\text{cl}}(q, p)$

$$w_f(X, \mu, \nu) = \int f_{\text{cl}}(q, p) \delta(\mu q + \nu p - X) dq dp. \quad (22)$$

It is called classical tomogram. The classical tomogram can be determined as the expectation value of a delta-function calculated with the help of the distribution function $f_{\text{cl}}(q, p)$ in the phase space

$$w_f(X, \mu, \nu) = \langle \delta(\mu q + \nu p - X) \rangle. \quad (23)$$

The classical tomogram is nonnegative function and it is normalized

$$\int w_f(X, \mu, \nu) dX = 1.$$

The physical meaning of the classical tomogram is that it is the probability density for the particle's coordinate X , which is measured in the phase-space reference frame subjected to the scaling of axes and subsequent rotation with respect to the original reference frame. In the same way as in the case of Fourier transformation, where information contained in the function is equivalent to information contained in its Fourier transform, information on the particle's state contained in

the distribution function $f_{\text{cl}}(q, p)$ is equivalent to information contained in its Radon transform – the classical tomogram $w_f(X, \mu, \nu)$. The Radon transformation is invertible

$$f_{\text{cl}}(q, p) = \frac{1}{4\pi^2} \int w_f(X, \mu, \nu) \exp[-i(\mu q + \nu p - X)] dX d\mu d\nu, \quad (24)$$

so the probability distribution function $f_{\text{cl}}(q, p)$ can be reconstructed in view of the classical tomogram $w_f(X, \mu, \nu)$.

The commutative star-product kernel $K(X, \mu, \nu, X_1, \mu_1, \nu_1, X_2, \mu_2, \nu_2)$ for two classical tomograms $w_{f1}(X_1, \mu_1, \nu_1)$ and $w_{f2}(X_2, \mu_2, \nu_2)$ is [9]

$$K_{\text{classic}}(X, \mu, \nu, X_1, \mu_1, \nu_1, X_2, \mu_2, \nu_2) = \frac{1}{(2\pi)^2} e^{i(X_1+X_2-X(\nu_1+\nu_2)/\nu)} \delta(\nu(\mu_1 + \mu_2) - \mu(\nu_1 + \nu_2)).$$

The relationship between tomographic star-product kernels in quantum and classical mechanics reads [8]

$$K_{\text{quant}}(X, \mu, \nu, X_1, \mu_1, \nu_1, X_2, \mu_2, \nu_2) = K_{\text{classic}}(X, \mu, \nu, X_1, \mu_1, \nu_1, X_2, \mu_2, \nu_2) e^{[i(\mu_2\nu_1 - \mu_1\nu_2)/2]}.$$

6 CONCLUSIONS

We reviewed two tomographic-probability approaches — symplectic tomography and photon-number tomography and showed that they are examples of the star-product quantization scheme. We mentioned that, in the quantum case, the state of the system can be described by the probability distribution function instead of the density operator. This provides the possibility to describe the objects of both the quantum and classical natures using the same objects – tomograms.

ACKNOWLEDGMENTS

The author thanks the Organizers of the Conference "Foundation of Probability and Physics 5" and especially Prof. A. Khrennikov for invitation and kind hospitality.

Список литературы

- [1] S. Mancini, V. I. Man'ko, and P. Tombesi, *Quantum Semiclass. Opt.* **7**, 615-624 (1995).
- [2] S. Mancini, V. I. Man'ko, and P. Tombesi, *Phys. Lett. A* **213**, 1-6 (1996).
- [3] S. Mancini, V. I. Man'ko, and P. Tombesi, *Europhys. Lett.* **37**, 79-84 (1997).
- [4] S. Wallentowitz and W. Vogel, *Phys. Rev. A* **53**, 4528-4533 (1996).
- [5] K. Banaszek and K. Wodkiewicz, *Phys. Rev. Lett.* **76**, 4344-4347 (1996).
- [6] O. V. Man'ko and V. I. Man'ko, *J. Russ. Laser Res.* **18**, 407-444 (1997).

- [7] O. V. Man'ko and V. I. Man'ko, *J. Russ. Laser Res.* **21**, 411-437 (2000).
- [8] O. V. Man'ko, V. I. Man'ko, and O. V. Pilyavets, *J. Russ. Laser Res.* **26**, 429-444 (2005).
- [9] O. V. Man'ko and V. I. Man'ko, *J. Russ. Laser Res.* **25**, 477-492 (2004).
- [10] O. V. Man'ko, V. I. Man'ko, and G. Marmo, *J. Phys. A* **35**, 699-720 (2002).
- [11] O. V. Man'ko, V. I. Man'ko, and G. Marmo, "Tomographic map within the framework of star-product quantization" in *Proceedings of the Second International Symposium on Quantum Theory and Symmetries (Krakow, Poland, 2001)*, edited by E. Kapuschik and A. Horzela, World Scientific, Singapore, 2001, pp. 126-132.
- [12] S. V. Kuznetsov and O.V. Man'ko, *Proc. SPIE* **5402**, 302-313 (2004).
- [13] O. V. Man'ko, *J. Russ. Laser Res.* **28**, 125-135 (2007).
- [14] O. V. Man'ko, V. I. Man'ko, G. Marmo, and P. Vitale, *Phys. Lett. A* **360**, 522-532 (2007).
- [15] J. Bertrand and P. Bertrand, *Found. Phys.* **17**, 397-405 (1987).
- [16] K. Vogel and H. Risken, *Phys. Rev. A* **40**, 2847-2849 (1989).
- [17] K. E. Cahill and R. J. Glauber, *Phys. Rev.* **177**, 1882-1902 (1969).
- [18] E. Wigner, *Phys. Rev.* **40**, 749-759 (1932).
- [19] O. V. Man'ko, "Photon-number tomogram for two-mode squeezed states" in *Proceedings of the 8th International Conference on Squeezed States and Uncertainty Relations*, edited by H. Moya-Cessa, et al., Rinton Press, 2003, pp. 254-261.
- [20] O. V. Man'ko and V. I. Man'ko, *J. Russ. Laser Res.* **24**, 497-506 (2003).
- [21] V. V. Dodonov, O. V. Man'ko, and V. I. Man'ko, *Phys. Rev. A* **49**, 2993-3001 (1994).